Noether's Theorem

- continuous symmetries of the Lagrangian
- automatic conservation laws from continuous symmetries

We've all heard mumblings how symmetries of space — translations, rotations — are responsible for the fundamental conservation laws — linear momentum, angular momentum. But how is this connection made precise? The answer lies in the Lagrangian and its invariance with respect to <u>continuous</u> <u>transformations</u>.

Noether's theorem provides a way of teasing out the conserved quantities directly from the Lagrangian, just from the mathematical transformation law. That turns out to be extremely useful in exotic spheres of physics (e.g. particle theory) where intuitive/geometric understanding of symmetries can be elusive!

Let's start with an example. Consider the Lagrangian for a particle freely moving in the plane (no potential energy). In polar coordinates:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

It was discovered only recently that this Lagrangian possesses a mysterious continuous symmetry, called "S-symmetry". The continuous symmetry is described by a continuous parameter S. When S=0 there is no transformation (the "identity") and for small S (= δS) the transformation rule of the coordinates has the form (next page)

S-symmetry transformation:

$$r \to r + \delta r \quad , \quad \theta \to \theta + \delta\theta$$

$$\delta r = \cos\theta \, \delta s$$

$$\delta\theta = -\frac{\sin\theta}{r} \delta s$$

Let's verify that in the limit of small δs this is indeed a "symmetry of the Lagrangian", i.e. it leaves L unchanged:

$$\dot{r} \longrightarrow \dot{r} + \delta\dot{r} = \dot{r} - (\sin\theta \cdot \dot{\theta})\delta s$$

$$\dot{\theta} \longrightarrow \dot{\theta} + \delta\dot{\theta} = \dot{\theta} - \left(\frac{\cos\theta}{r}\dot{\theta}\right)\delta s + \left(\frac{\sin\theta}{r^2}\dot{r}\right)\delta s$$

$$r \longrightarrow r + \cos\theta \, \delta s$$

We keep changes to L only to order δs^1:

$$\dot{r}^2 + r^2\dot{\theta}^2 \longrightarrow (\dot{r}+\delta\dot{r})^2 + (r+\delta r)^2(\dot{\theta}+\delta\dot{\theta})^2$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{r}\delta\dot{r} + 2r\delta r\dot{\theta}^2$$

$$+ r^2 2\dot{\theta}\delta\dot{\theta} + \mathcal{O}(\delta s^2)$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{r}(-\sin\theta\,\dot{\theta}\,\delta s)$$

$$+ 2r(\cos\theta\,\delta s)\dot{\theta}^2 + 2r^2\dot{\theta}\left(-\frac{\cos\theta}{r}\dot{\theta} + \frac{\sin\theta}{r^2}\dot{r}\right)\delta s$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + \mathcal{O}(\delta s^2)$$

Since the coefficient of δs^1 is zero, we say the invariance is exact in the limit of small δs.

Let's formalize the notion of "invariance with respect to a continuous symmetry" in the case of a general Lagrangian with generalized coordinates $q_1, q_2, \dots$

First introduce the transformed coordinates

$$Q_1(s),\ Q_2(s),\ \dots$$

which are parametrized by our continuous symmetry parameter s.

In our earlier example,

$$Q_1(\delta s) = r + \cos\theta \cdot \delta s$$

$$Q_2(\delta s) = \theta - \frac{\sin\theta}{r}\,\delta s.$$

At $s=0$ we recover the un-transformed coordinates $Q_1(0) = q_1$ etc.

The statement that the Lagrangian is invariant to first order in the continuous parameter s is the condition

$$\frac{d}{ds} L\left(Q_1(s), Q_2(s), \ldots, \mathring{Q}_1(s), \mathring{Q}_2(s), \ldots\right)\Bigg|_{s=0} = 0$$

(the second term of the Taylor series in s, about $s=0$, is zero)

Let's proceed to evaluate the derivative above, by applying the multi-variable chain rule:

$$0 = \sum_{i=1}^{N} \left(\frac{\partial L}{\partial Q_i} \frac{dQ_i}{ds} + \frac{\partial L}{\partial \mathring{Q}_i} \frac{d\mathring{Q}_i}{ds} \right)\Bigg|_{s=0}$$

We can use the fact that the Euler-Lagrange equation still holds (for any fixed s):

$$\frac{\partial L}{\partial Q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i}$$

$$0 = \sum_{i=1}^{N}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i}\right)\frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i}\left(\frac{d}{dt}\frac{dQ_i}{ds}\right)\right)\Bigg|_{s=0}$$

Now apply the product rule of differentiation:

$$0 = \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{\partial L}{\partial \dot{Q}_i}\frac{dQ_i}{ds}\right)\Bigg|_{s=0}$$

Finally, we use the definition of <u>conjugate momentum</u>:

$$\left.\frac{\partial L}{\partial \dot{Q}_i}\right|_{s=0} = \frac{\partial L}{\partial \dot{q}_i} = P_i$$

$$0 = \frac{d}{dt}\underbrace{\left(\sum_{i=1}^{N} P_i \left.\frac{dQ_i}{ds}\right|_{s=0} \right)}_{I}$$

We see that the quantity I above is <u>conserved</u>.

Let's evaluate I for the mysterious continuous symmetry of a particle moving in the plane:

$$P_r = m\dot{r} \qquad \left.\frac{dQ_r}{ds}\right|_{s=0} = \cos\theta$$

$$P_\theta = mr^2\dot{\theta} \qquad \left.\frac{dQ_\theta}{ds}\right|_{s=0} = -\frac{\sin\theta}{r}$$

$$I = m\dot{r}\cdot\cos\theta + mr^2\dot{\theta}\left(-\frac{\sin\theta}{r}\right)$$

$$= m(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})$$

Question 1 : Is this really a new conserved quantity?

$$x = r\cos\theta$$

$$\Longrightarrow \quad \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$I = m\dot{x} = p_x$$

Question 2 : Describe the mysterious "S-symmetry" in simpler terms.

A: Continuous translations (shifts) of the x-coordinate:

$$x \longrightarrow x + S$$

(Its identity was obscured by polar coordinates.)